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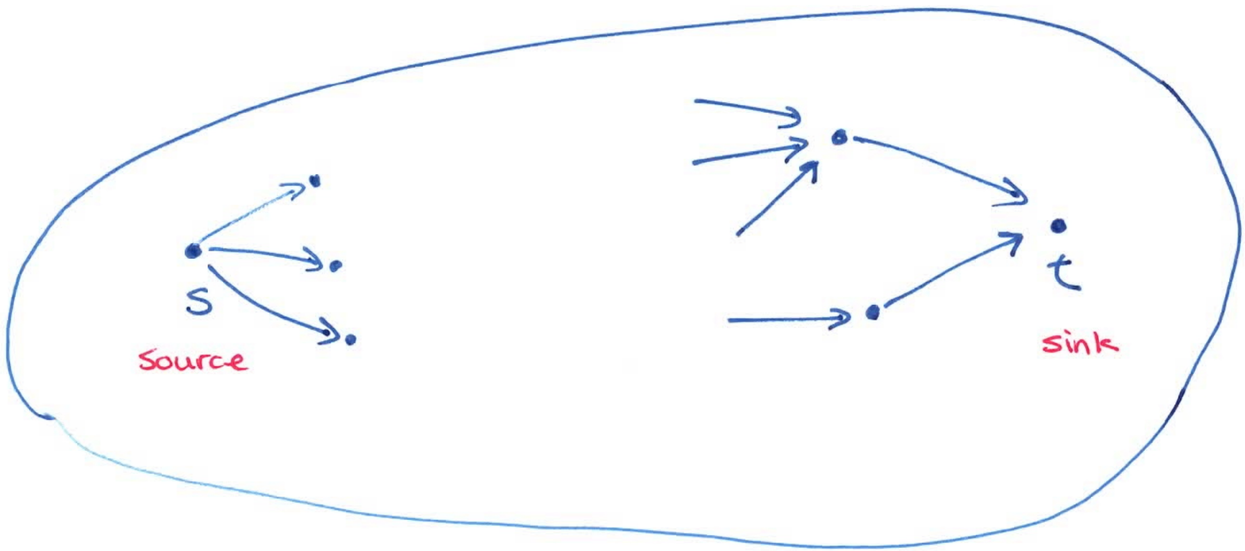
FLOW NETWORKS

- graphs with capacities on edges
- algorithms can be complicated
- many varieties, we'll study simplest form
- recast problems as flow problems

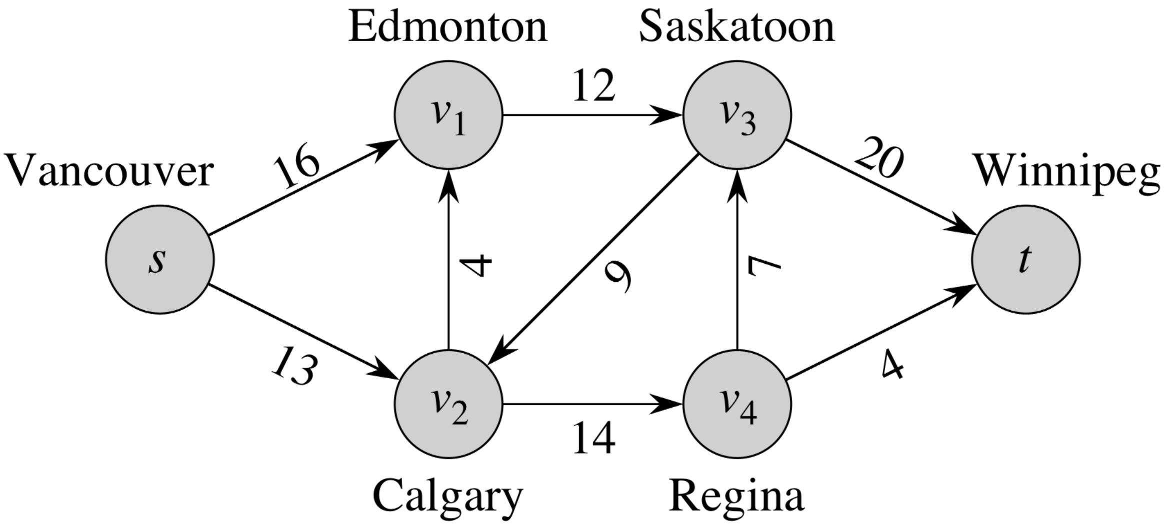
Introduction to Flow Networks

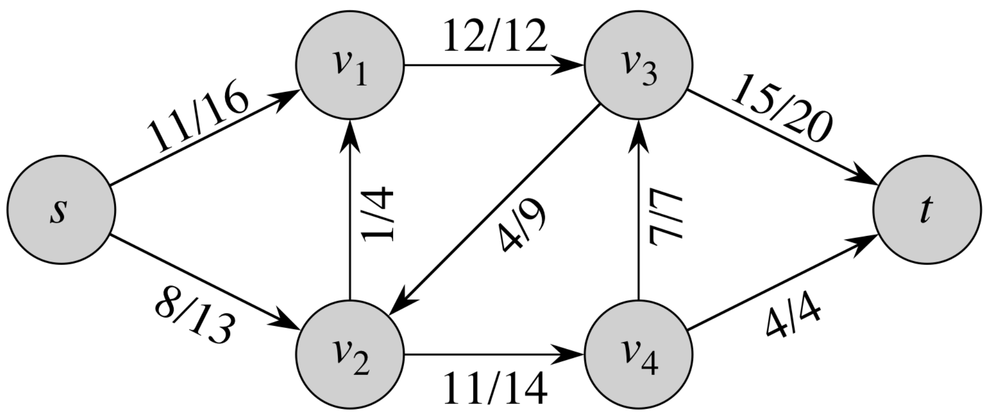


Suppose we want to move n units from source s to sink t.

Suppose edge cannot carry all n units.

Cannot send all n units through the same path.





Throughput
= 19 units
= $|f|$
= flow value

$G = (V, E)$ a directed graph

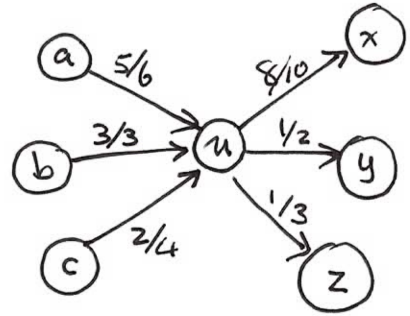
$c : E \to \mathbb{R}$ assigns a non-negative capacity to each edge

$f : E \to \mathbb{R}$ assigns a flow through each edge

Legal flow:

① flow conservation

except source & sink → total flow in = total flow out



② $f(u,v) \leq c(u,v)$

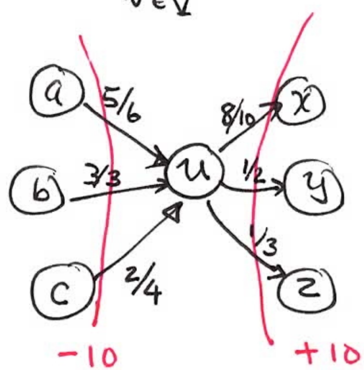
Skew symmetry: require for every edge (u,v)

$$f(u,v) = -f(v,u)$$

not every textbook follows this convention.

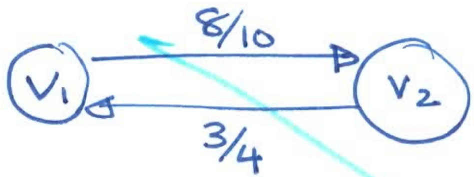
We can restate flow conservation as:

for all $u \in V$, $\displaystyle\sum_{v \in V} f(u,v) = 0.$

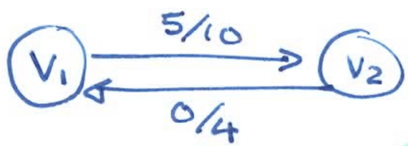


sum=0

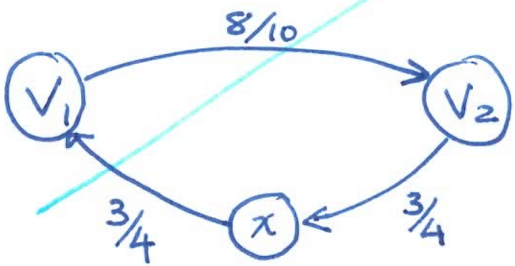
Justifying Skew Symmetry



not allowed to have $f(v_1, v_2) = 8$ & $f(v_2, v_1) = 3$



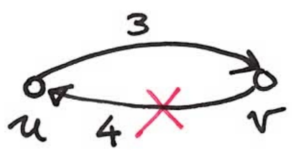
equivalent for flow problems



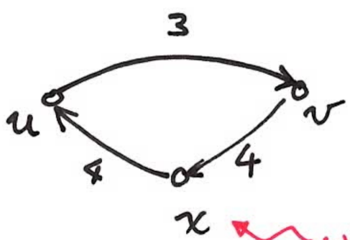
if you really, really, really want flow in both directions, add a dummy vertex

③ No anti-parallel edges:

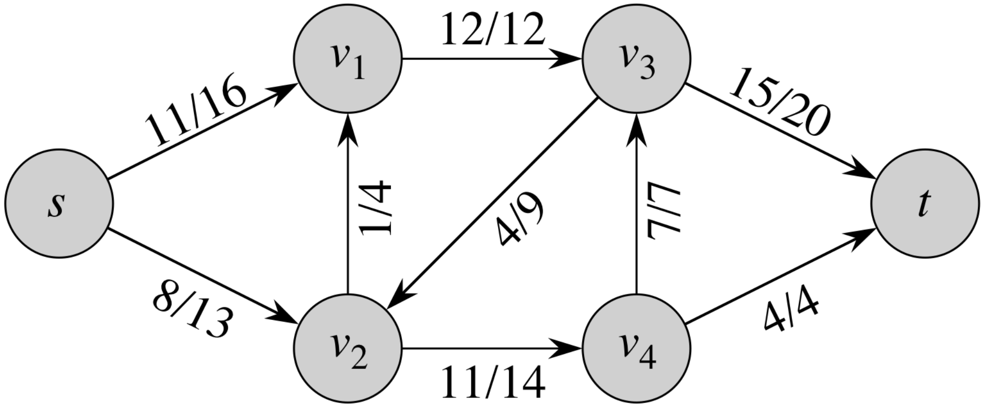
if $(u,v) \in E$, then $(v,u) \notin E$.

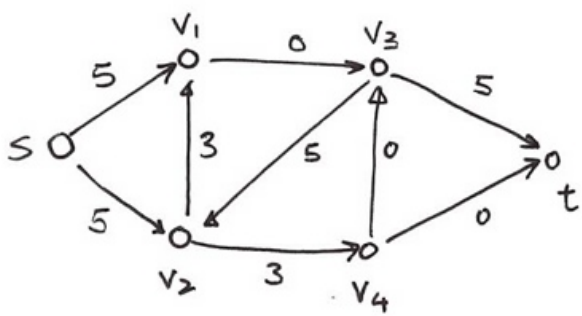


We can add a new vertex, if needed

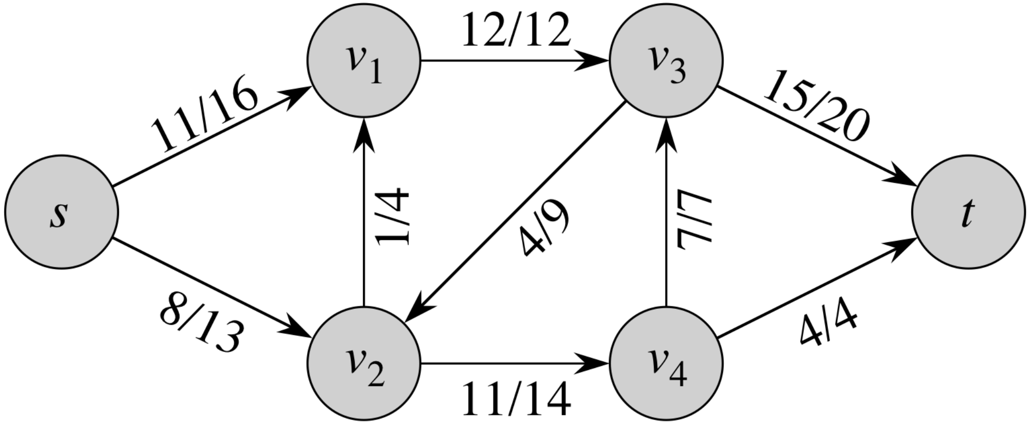


Is this flow the maximum?

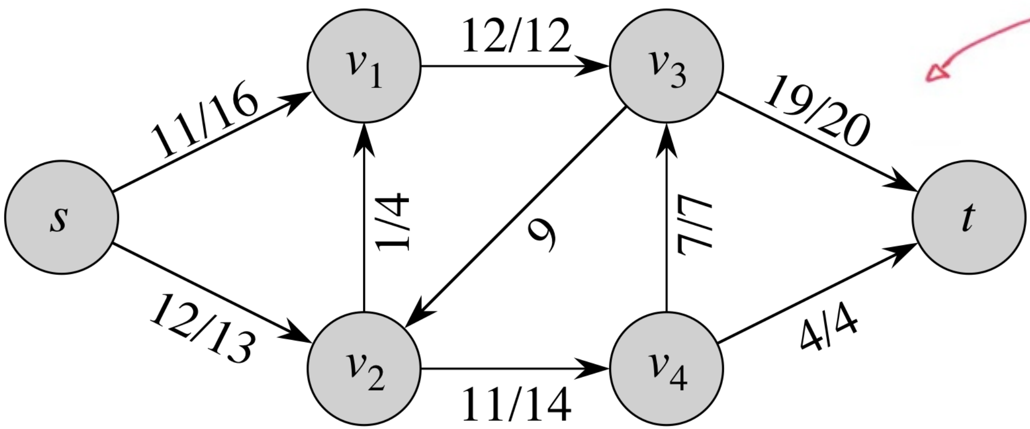




← remaining capacities



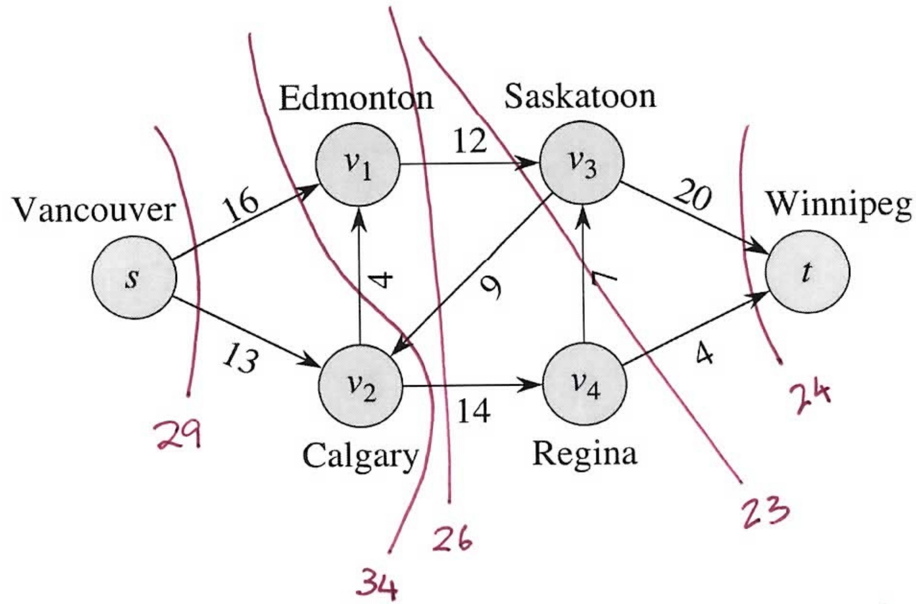
(a)



Is this max?

why?

(c)



Minimum cut = 23

Defn: a *cut* in a flow network is a partition of the vertices V into S and T such that $s \in S$ & $t \in T$.

means that
$S \cap T = \emptyset$
$S \cup T = V$

Defn: the capacity of a cut (S,T) is:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

sum of capacities of edges that cross the cut from S side to T side

Intuitively, for any flow f, $|f| \leq c(S,T)$ for <u>any</u> cut (S,T).

Residual Graphs

means left overs

Given $G=(V,E)$, $c: E \rightarrow \mathbb{R}$ and legal flow $f: E \rightarrow \mathbb{R}$

Residual graph $G_f = (V, E_f)$ with capacity $c_f : E_f \rightarrow \mathbb{R}$

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

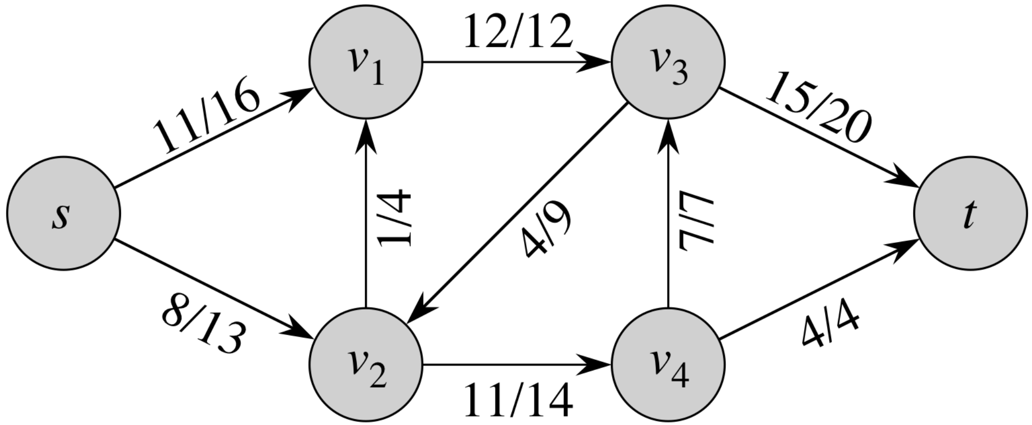
anti-parallel edges not allowed in E, but are allowed in G_f.

$c(u,v) = 5 \quad f(u,v) = 3$

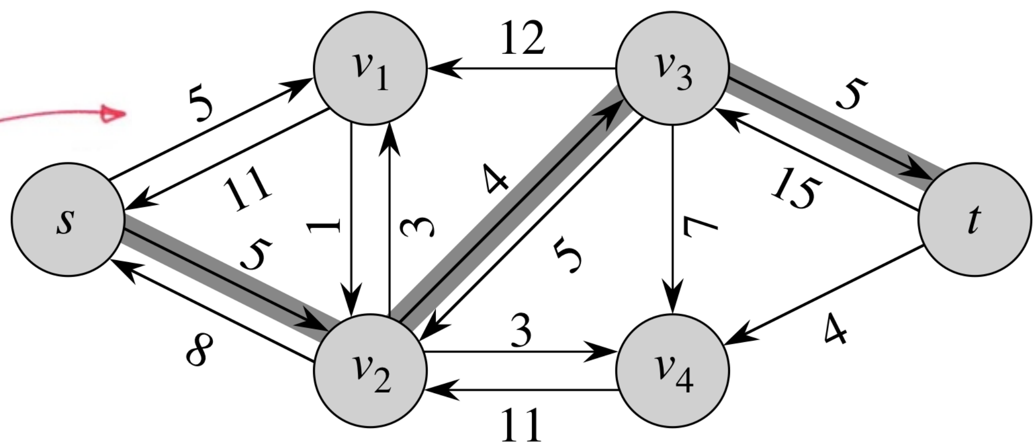
$c_f(u,v) = 2$

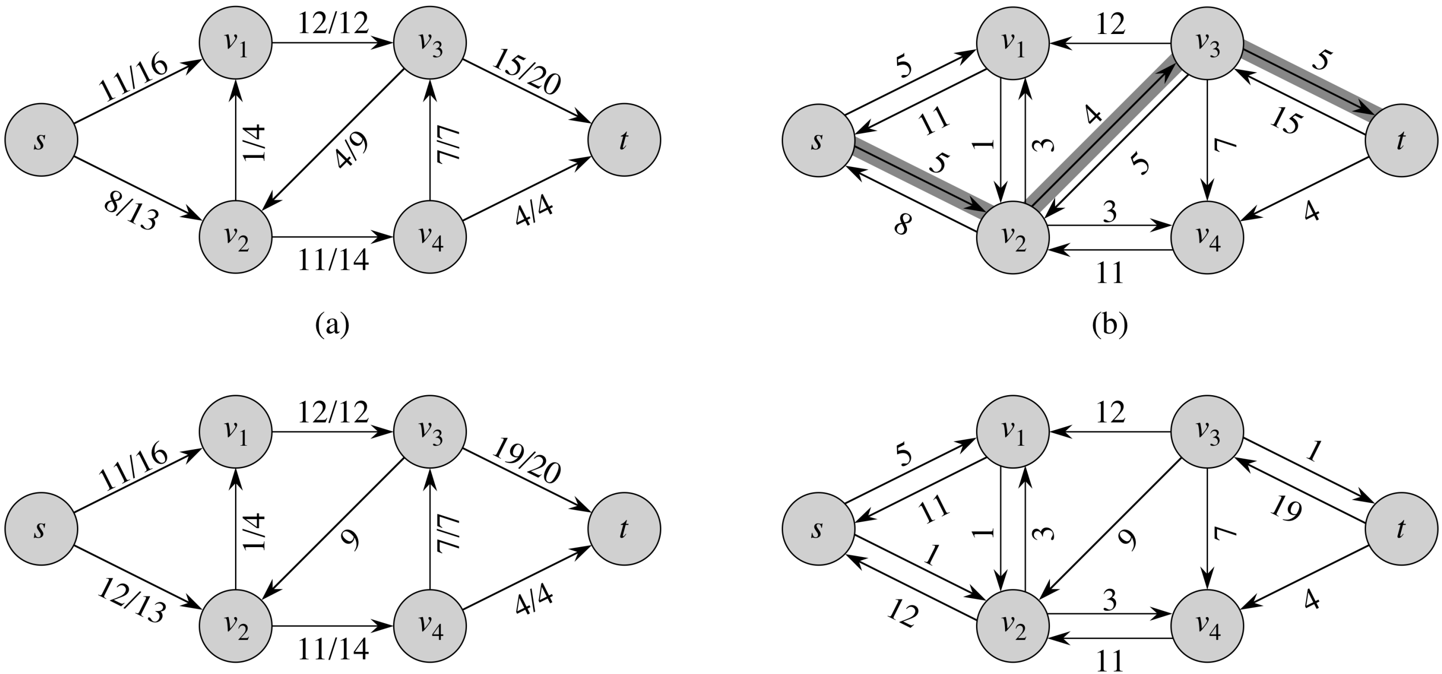
$c_f(v,u) = 3$

$$E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$$



a residual graph lets us find an augmenting path





(a) (b)

(c) (d)

no more augmenting paths in (d).

Augmenting paths

A path p from s to t in a residual graph G_f is called an augmenting path.

max units we can push thru the path

$$c_f(p) = \min\{ c_f(u,v) \mid (u,v) \text{ is an edge in } p \}$$

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \\ 0 & \text{o.w.} \end{cases}$$

use this to improve flow f.

$$f^+ = (f \uparrow f_p)$$

augment f by flow in f_p

$$f^+(u,v) = \begin{cases} f(u,v) + f_p(u,v) & \text{if } (u,v) \in E \ \& \ f_p(u,v) \geq 0 \\ f(u,v) - f_p(v,u) & \text{if } (u,v) \in E \ \& \ f_p(v,u) > 0 \\ 0 & \text{if } (u,v) \notin E \end{cases}$$

Claim: if f is a legal flow and f_P is an augmenting path in G_f, then

$$f^+ = f \uparrow f_P$$

is a legal flow in G.

Pf:

Just check $f^+(u,v) \leq c(u,v)$ ← check cases for $f_P(u,v) > 0$ & $f_P(v,u) > 0$

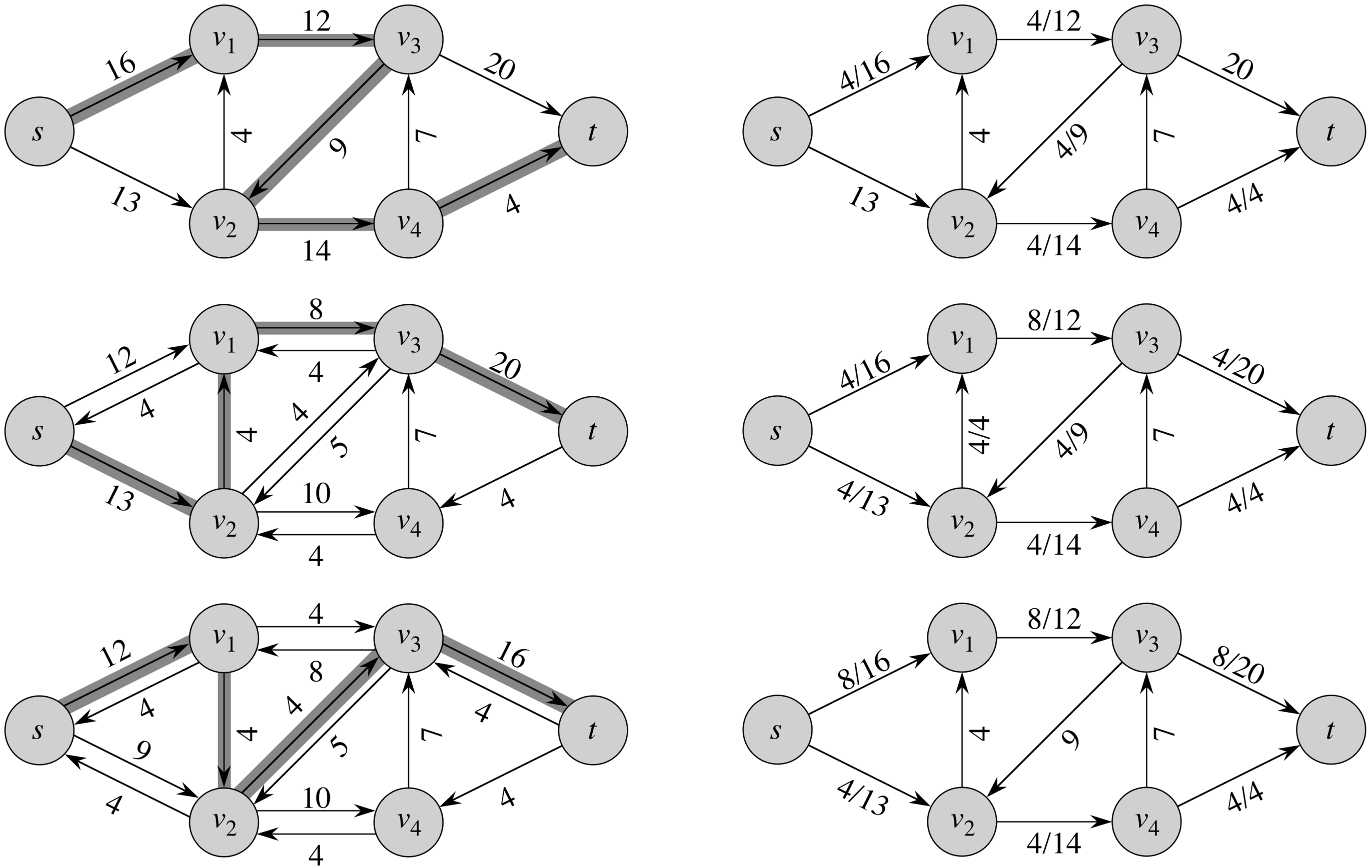
and that flow in f^+ is conserved.

$\boxtimes$

Ford-Fulkerson method

1. f = zero flow
2. Construct residual graph G_f
3. Find an augmenting path p & construct f_p
4. Let $f := f \uparrow f_p$

Repeat until no augmenting paths are found


s
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8/20

(a)

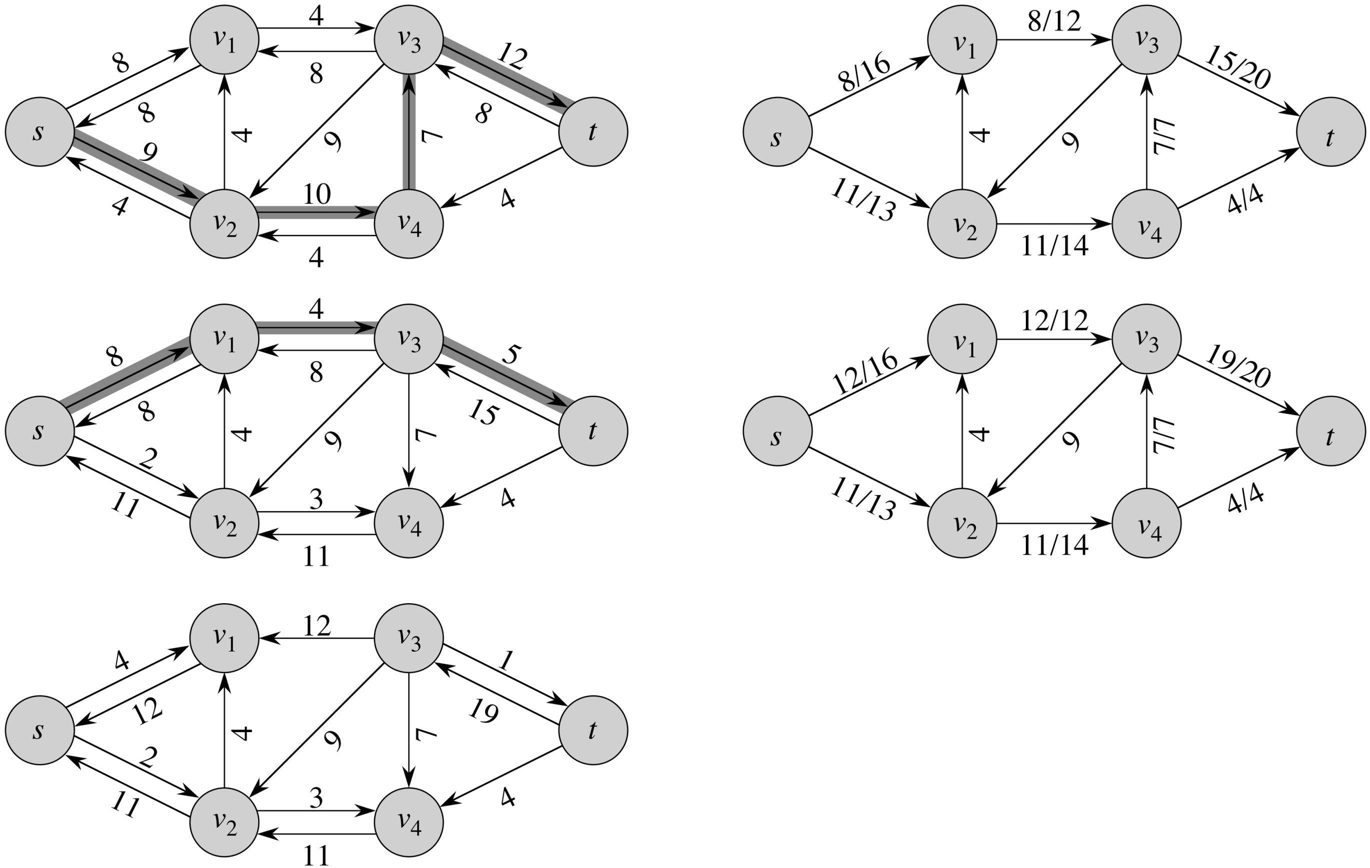
(b)

(c)

(d)

(e)

(f)



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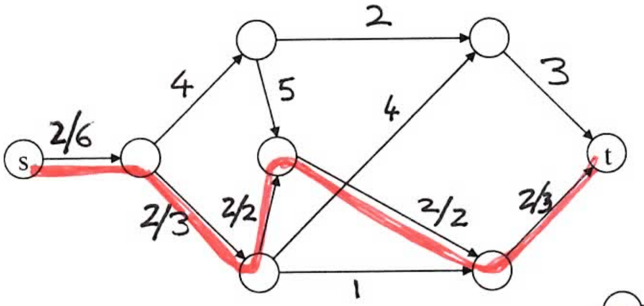
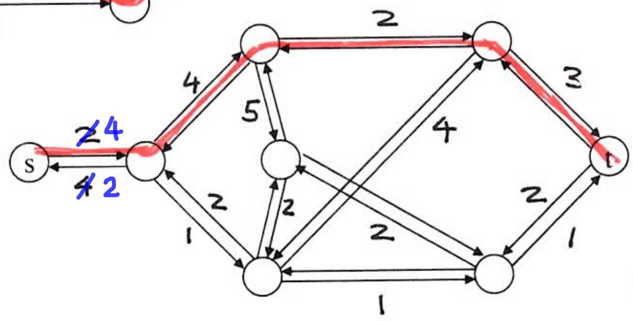
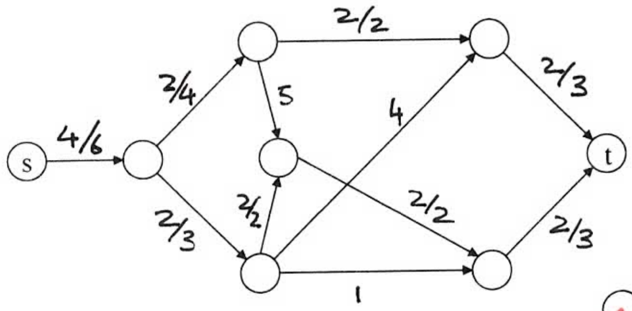


capacity of augmenting path

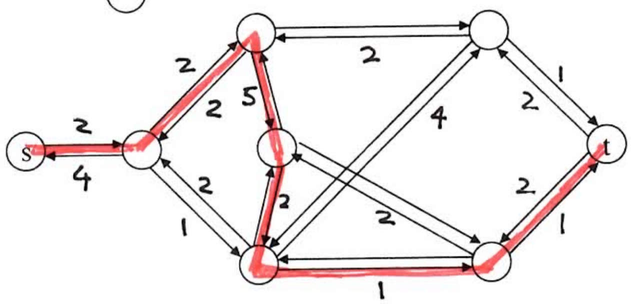
$c_p = 2$



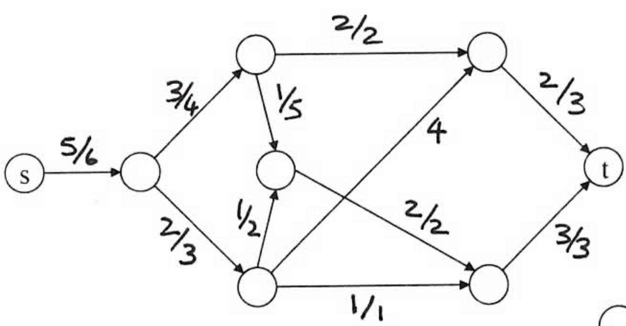
$c_p = 2$

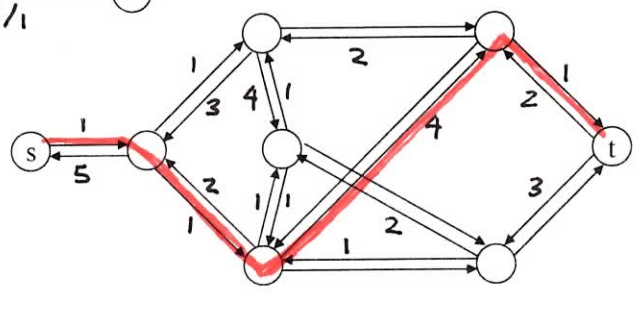


flow so far gives this residual graph

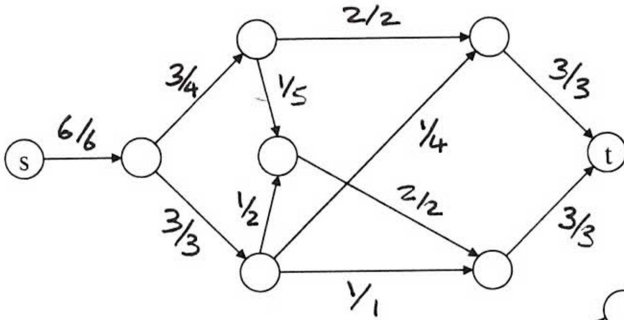


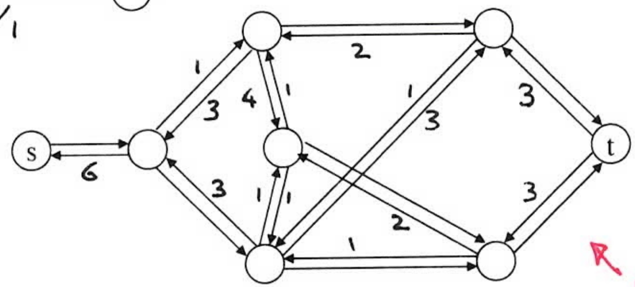
$c_p = 1$





$c_p = 1$





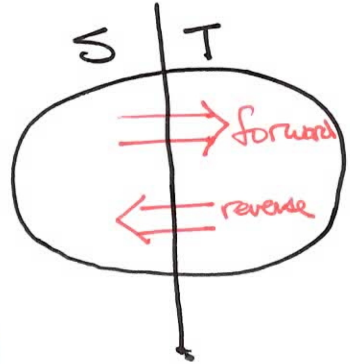
no augmenting paths

Max Flow Min Cut Theorem

Let f be a legal flow in a flow network $G=(V,E)$. Then the following are equivalent:

1. f is a maximum flow in G
2. G_f has no augmenting paths.
3. $|f|$ = cut capacity of some cut (S,T)

Cuts, flows & capacities



For any cut (S,T), we define

$$f(S,T) = \text{net flow across } (S,T)$$

$$= \underbrace{\sum_{u \in S} \sum_{v \in T} f(u,v)}_{\text{forward flow}} - \underbrace{\sum_{u \in S} \sum_{v \in T} f(v,u)}_{\text{reverse flow}}$$

$$c(S,T) = \text{capacity of } (S,T)$$

$$= \sum_{u \in S} \sum_{v \in T} c(u,v)$$

Lemma for any cut (S,T), $f(S,T) = |f|$. ← flow value

intuition: net flow is the same, no matter how you cut it.

Pf:

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) \quad \text{by definition.}$$

Flow conservation gives us:

$$\underbrace{\sum_{v \in V} f(u,v)}_{\text{flow out of } u} - \underbrace{\sum_{v \in V} f(v,u)}_{\text{flow into } u} = 0 \quad \text{for all } u \notin \{s,t\}$$

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S'} \Big(\overbrace{\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u)}^{=0\text{, by flow conservation}} \Big)$$

where $S' = S - \{s\}$

$$|f| = \sum_{v\in V} f(s,v) - \sum_{v\in V} f(v,s) + \sum_{u\in S'} \sum_{v\in V} f(u,v) - \sum_{u\in S'} \sum_{v\in V} f(v,u)$$

switch order of summation

$$= \sum_{v\in V} f(s,v) - \sum_{v\in V} f(v,s) + \sum_{v\in V} \sum_{u\in S'} f(u,v) - \sum_{v\in V} \sum_{u\in S'} f(v,u)$$

regroup

$$= \sum_{v\in V} \left(f(s,v) + \sum_{u\in S'} f(u,v) \right) - \sum_{v\in V} \left(f(v,s) + \sum_{u\in S'} f(v,u) \right)$$

$$= \sum_{v\in V} \sum_{u\in S} f(u,v) - \sum_{v\in V} \sum_{u\in S} f(v,u)$$

because $S' = S - \{s\}$

$$|f| = \sum_{v\in V}\sum_{u\in S} f(u,v) - \sum_{v\in V}\sum_{u\in S} f(v,u)$$

Split V into S & T

recall $V = S\cup T$

$S\cap T = \emptyset$

$$= \sum_{v\in S}\sum_{u\in S} f(u,v) + \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u) - \sum_{v\in T}\sum_{u\in S} f(v,u)$$

regroup

$$= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) + \left(\sum_{v\in S}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u)\right)$$

swith order of summation

rename variables

$$= \underbrace{\sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u)}_{\text{by defn}} + \left(\sum_{x\in S}\sum_{y\in S} f(y,x) - \sum_{y\in S}\sum_{x\in S} f(y,x)\right)$$

order of sum.

$$= f(S,T) \qquad + \underbrace{\left(\sum_{x\in S}\sum_{y\in S} f(y,x) - \sum_{x\in S}\sum_{y\in S} f(y,x)\right)}_{=0}$$

$$= f(S,T)$$

$\boxtimes$

Corollary: Let (S,T) be any cut, then $|f| \le c(S,T)$.

Pf: $|f| = f(S,T)$ from previous lemma

$$= \sum_{u\in S}\sum_{v\in T} f(u,v) - \underbrace{\sum_{u\in S}\sum_{v\in T} f(v,u)}_{\text{non negative}} \quad \text{by def'n}$$

$$\le \sum_{u\in S}\sum_{v\in T} f(u,v) \quad \text{since } f(v,u) \ge 0$$

$$\le \sum_{u\in S}\sum_{v\in T} c(u,v) \quad \text{since } f(u,v) \le c(u,v)$$

$$= c(S,T) \qquad \boxtimes$$

we will show they are actually equal

Then, value of maximum flow $\le$ capacity of minimum cut.

Proof of Max Flow Min Cut Theorem

Recap:

Theorem: Let f be a flow in a flow network $G=(V,E)$

Then the following are equivalent

1. f is a max flow in G
2. G_f has no augmenting paths
3. $|f| =$ cut capacity of some cut (S,T)

Defn: the cut capacity of $(S,T) = c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$

= sum of capacities of edges that cross from S to T.

Strategy: Show ① $\Rightarrow$ ② [actually $\neg$② $\Rightarrow$ $\neg$①]

② $\Rightarrow$ ③

③ $\Rightarrow$ ①

① $\Rightarrow$ ②

f is a max flow in G $\Rightarrow$ G_f has no augmenting paths

Obvious.

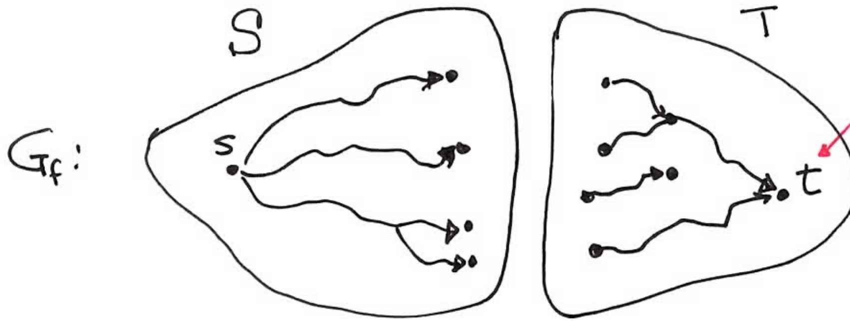
If G_f has an augmenting path p, then $f' = f \uparrow f_p$ is a legal flow s.t. $|f'| > |f|$.

This contradicts the assumption that f is a max flow.

② ⇒ ③

If G_f has no augmenting paths, then $|f|$ = cut capacity of some cut (S,T)

Suppose G_f has no augmenting paths



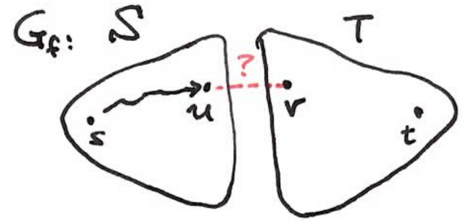
cannot reach t from s since G_f has no aug. paths.

residual graph, not original graph

Let $S = \{u \in V \mid$ there is a path from s to u in $G_f\}$

$T = V - S$ ← can include vertices that cannot reach t.

(S,T) is a cut since $s \in S$ and $t \in T$.



Consider $u \in S$ and $v \in T$.

- A: if $(u,v) \in E$, then we must have $f(u,v) = c(u,v)$. O.w. G_f will have edge (u,v) & $s \rightsquigarrow v$. $\Rightarrow\Leftarrow$
- B: if $(v,u) \in E$, then we must have $f(v,u) = 0$. Otherwise, we can send units back to v and $(u,v) \in E_f$. $\Rightarrow\Leftarrow$
- C: if $(u,v) \notin E$ and $(v,u) \notin E$, then $f(u,v) = 0$ & $f(v,u) = 0$.

Then, $$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} \cancelto{0}{f(v,u)} \text{ by B}$$

by A & C

$$= \sum_{u \in S} \sum_{v \in T} c(u,v) \qquad = c(S,T) \text{ by defn}$$

By previous lemma, $|f| = f(S,T)$, so $|f| = c(S,T)$

③ $\Rightarrow$ ①

From previous Corollary, we know that $|f'| \leq c(S,T)$, for any flow f'.

Now, if $|f| = c(S,T)$ for a particular flow f,

then for any flow f', $|f'| \leq c(S,T) = |f|$.

Thus, f is a maximum flow.

$\boxtimes$